*Journal of Engineering Mathematics* **46:** 355–376, 2003.


# Equilibrium of an elastic finite cylinder: Filon's problem revisited

V. V. MELESHKO
*Kiev National Taras Shevchenko University, 01033 Kiev, Ukraine. e-mail: meleshko@univ.kiev.ua*

Received 11 June 2002; accepted in revised form 15 January 2003

**Abstract.** This paper deals with an analytical solution of the axisymmetric boundary-value problem of the theory of elasticity for a finite circular cylinder with free ends and arbitrary loaded curved surface. The object of this paper is to employ the method of superposition to obtain accurate values of the stress field near the boundaries. The classical Filon (1902) problem of uniformly distributed tangential load applied along two rings at the curved surface is addressed in full detail. The distribution of stresses along some typical sections of the cylinder are shown graphically.

**Key words:** finite elastic cylinder, Filon's problem, biharmonic equation, method of superposition

## 1. Introduction

This paper considers the axisymmetric distribution of stresses and displacements in a finite elastic cylinder under nonuniform and discontinuous loading applied at its curved surface. An analytical method is presented and relevant results are discussed in an historical perspective.

The problem of equilibrium of an elastic cylinder of finite length subjected to a surface load on its surface is one of the oldest problems in the theory of elasticity.[1] Elementary solutions for the extension and compression of circular elastic bars (short cylinders) traditionally assume that they are subjected to a normal tension or pressure uniformly distributed across the ends. It follows that the extension or compression are transmitted throughout the bar without change. Such applied loadings, however, do not usually occur in practice.

In an extensive memoir [1] submitted to the Royal Society on May 20, 1901 (see also a detailed abstract [2]) Filon addressed several problems concerning distributions of stresses and displacements in a circular elastic finite cylinder under some axisymmetric systems of surface loadings. In particular, Filon was interested to find out how the results obtained for such a theoretical system of loading are modified, if at all, when we consider applied external stresses, which give a closer representation of practical mechanical conditions, and considered in great details the typical case when shearing pull loading is applied at the curved surface of the cylinder. He argued [1, p. 148]

[1] Already in 1846 the competition for the *Grand Prix de Mathématiques* of the Academy of Science of Paris for a solution of this problem was announced for the year 1848. According to the announcement published in *Comptes rendus des séances de l'Académie des Sciences*, **22**, 768–769 (séance du 11 mai 1846) the condition for the award was: '*Trouver les intégrales des équations de l'équilibre intérieur d'un corps solide élastique et homogène dont toutes les dimensions sont finies, par exemple d'un parallélipipède ou d'un cylindre droit, en supposant connues les pressions ou tractions inégales exercées aux différents points de sa surface*. Le prix consistera en une médaille d'or de la valeur de *troix mille francs*.' There were no entries and the committee initially consisted of Arago, Cauchy, Lamé, Sturm and Liouville. They suggested this topic (along with the last Fermat theorem!) then two times for the years 1853 and 1857. It had been prolonged for the year 1861, but already in 1858, by a committee consisting of Liouville, Lamé, Duhamel and Bertrand, was changed into another question.

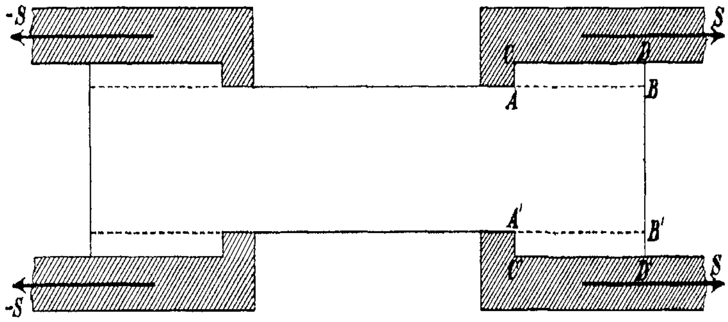


*Figure 1.* Filon's [1] sketch of a specimen in a testing machine.

> This will give us valuable information about a system of stress which often occurs in practice, in testing machines, for example, in which a specimen is pulled apart by means of pressures applied to the inner rims of projecting collars (see Figure 1). The shaded parts of the figure represent the 'grips', and if S be the total pull applied, this is transmitted to the test piece by means of presure applied along CA, C′A′. Now consider the thinner cylinder in the middle ideally produced inside the thicker ends. It is in equilibrium under the stresses, radial and tangential, between the inner core and the hollow cylinder produced by the revolution of ABCD.
>
> But what are these radial and tangential stresses? If we consider the equilibrium of the outer hollow cylinder only, we see that the resultant of the stresses across AB, A′B′ must exactly balance the pull S, however applied. The radial stress will probably be small, as it has no external traction to balance, and the longitudinal shears are therefore equivalent to S. Thus the thin cylinder inside is really stretched, not by normal traction over the flat ends, but by longitudinal shears over the curved surface, and a careful investigation will show that, in every *practical* case, extension is obtained by the application of an axial shear to the curved surface of the cylinder, *never* of tractions to the flat ends. The general effects of such a distribution appear, therefore, of great practical interest.

The analytical solution of the equations of elasticity in cylindrical coordinates was chosen in the typical form of ordinary Fourier series on the complete systems of trigonometric functions of the axial coordinate. The arbitrary coefficients entering into functions of the radial coordinate are determined by comparison with the coefficients of the Fourier series which express the applied stresses at the curved boundary.

Filon pointed out that this method is not new, for it had already been indicated by Lamé and Clapeyron [3], (that time the *colonels du génie au service de Russie*), and he provided a short review of previous studies by Pochhammer [4] and Chree [5]. In an additional note Filon [1, p. 151] mentioned that the eigenfunctions-expansion approach to solve the problem suggested by Schiff [6] leads to certain transcendental equations and non-orthogonal systems of functions that essentially complicates the solution from a numerical point of view.

Filon performed extensive numerical calculations for a cylinder with a length that is about three times greater than the radius loaded in such a way that the two rings of uniform shear each extend over one-sixth of the length and are at equal distances from the mid-section and the two ends, and presented the results both in several tables and diagrams that show distributions of the displacements and stresses. His table for the axial stress data has often been reproduced, see, for example, [7–10]. Filon claimed that a self-equilibrating system of radial shears at the flat ends will have little or no effect at points at some distance from the flat ends, but since then there have been no attempts to check this statement.

The main object of this paper is to address anew the Filon problem with special attention paid to the stress field in the cylinder near its rims. We use an analytical solution of the problem obtained by the method of superposition. This method for axisymmetric problems of an elastic equilibrium of a finite cylinder was first suggested by Purser [11], but that paper has gone almost unnoticed, except for the study by Pickett [12]. The interest in the method of superposition was revived only in the fifties of the twentieth century when almost simultaneously two papers [13, 14] were published. By means of this approach Grinchenko [15, 16] considered thermal and centrifugal stresses in a finite elastic cylinder. These results were partially reproduced in [17]. A detailed survey of studies based upon the method of superposition can be found in [18].

The paper is organized as follows: the formulation of the problem is outlined in Section 2. The analytical method of superposition is described in Section 3, along with theoretical considerations about the stress field due to the discontinuous tangential load at the surface of the cylinder. Next, Section 4 describes numerical results concerning the distribution of stresses and displacements in comparison with Filon's example. Finally, some conclusions are given in Section 5.

## 2. Statement of axisymmetric problem for an elastic cylinder

Let us consider a circular finite elastic cylinder $0 \le r \le a$, $0 \le \phi \le 2\pi$, $-c \le z \le c$ (where $r, \phi, z$ being the usual cylindrical coordinates) which is subjected to an axisymmetric system of normal radial pressures and of tangential shears all over the curved surface; the plane ends are free from loadings.

If $u$ and $w$ denote the radial and longitudinal components, respectively, of the displacement vector, and they are independent of $\phi$, we have two Lamé equations

$$(\lambda + 2\mu)\,\frac{\partial^2 u}{\partial r^2} + (\lambda + 2\mu)\,\frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \mu\frac{\partial^2 u}{\partial z^2} + (\lambda + \mu)\frac{\partial^2 w}{\partial r \partial z} = 0\,, \tag{1}$$

$$(\lambda + \mu)\,\left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\,\frac{\partial u}{\partial z}\right) + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\,\frac{\partial w}{\partial r}\right) + (\lambda + 2\mu)\,\frac{\partial^2 w}{\partial z^2} = 0\,,$$

where $\lambda$ and $\mu$ are the elastic modulae of Lamé.

The stress components are related to the displacements by means of Hooke's law and the Cauchy relations

$$\begin{aligned} \widehat{rr} &= (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\frac{u}{r} + \lambda\frac{\partial w}{\partial z}\,, \quad \widehat{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\,, \\ \widehat{zz} &= (\lambda + 2\mu)\frac{\partial w}{\partial z} + \lambda\frac{u}{r} + \lambda\frac{\partial u}{\partial r}\,, \quad \widehat{\phi\phi} = (\lambda + 2\mu)\frac{u}{r} + \lambda\frac{\partial u}{\partial r} + \lambda\frac{\partial w}{\partial z}\,. \end{aligned} \tag{2}$$

Here and in what follows $\widehat{st}$ denotes the stress, parallel to d$s$, across an element of surface perpendicular to d$t$, $s$ and $t$ standing for any two of the letters $r$, $\phi$, $z$.

We will construct an analytical solution of a problem that is symmetric with respect to the plane $z = 0$ concerning the equilibrium of an elastic cylinder under the loadings

$$
\begin{aligned}
&\widehat{rr} = p(z)\,, \qquad \widehat{rz} = s(z) \qquad \text{at } \ r = a\,, \\
&\widehat{zz} = 0\,, \qquad\quad \widehat{rz} = 0 \qquad\quad \text{at } \ z = \pm c\,,
\end{aligned}
\tag{3}
$$

where $p(z)$ and $s(z)$ are given even and odd functions of $z$, respectively, with $s(c) = 0$.

Below we will consider two types of loadings: smooth normal and tangential loads and a discontinuous tangential loading. In particular, we will address the special case (which was considered in detail by Filon [1]) when the normal pressure is zero throughout, while the tangent loading $s(z)$ represents a constant axial shear $S$ acting over two equal rings on the curved surface of the cylinder, so that

$$
p(z) = 0\,, \qquad s(z) = \begin{cases} 0, & |z| < c_1 \ |z| > c_2 \\ S, & c_1 \le z \le c_2 \\ -S, & -c_2 \le z \le -c_1 \end{cases}
\tag{4}
$$

with $0 < c_1 < c_2 < c$. Important cases of discontinuous normal loading, concentrated normal or tangential forces acting at the curved surface can be considered either in a similar way or be obtained by the obvious limits from the previous cases.

## 3. Method of solution of the problem

The analytical solution of the boundary-value problem (1), (3) for the elastic cylinder consists of two steps. First, we need to construct the solution of the Lamé equations (1) in cylindrical coordinates. Second, we need to satisfy all boundary conditions (3) by means of a properly chosen representation for the displacement vector. As a rule, the second step is a far more difficult task in the whole solution process.

### 3.1. Construction of the solution of the Lamé equations

There exist several approaches to solve Equations (1) in cylindrical coordinates. We mention here only two of them.

Filon [1] established that both $\partial u/\partial z$ and $\partial w/\partial r$ satisfy the partial differential equation

$$
(\vartheta^2 + \mathrm{D}^2)^2\, y = 0\,,
\tag{5}
$$

with

$$
\vartheta^2 f = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rf)}{\partial r}\right), \qquad \mathrm{D}^2 f = \frac{\partial^2 f}{\partial z^2}
$$

and presented the solutions of these equations in terms of combinations of the trigonometric and Bessel functions.

Love [19, Section 188], established that for any axisymmetrical strain in an elastic body of revolution the displacements and the stress components can be expressed in terms of a single function $\chi$ as

$$
u = -\frac{1+\sigma}{E}\frac{\partial^2\chi}{\partial r\partial z}\,, \qquad w = \frac{1+\sigma}{E}\left\{(1-2\sigma)\nabla^2\chi + \frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r}\right\},
\tag{6}
$$

$$\widehat{rr} = \frac{\partial}{\partial z}\left\{\sigma\nabla^2\chi - \frac{\partial^2\chi}{\partial r^2}\right\}, \quad \widehat{rz} = \frac{\partial}{\partial r}\left\{(1-\sigma)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right\},$$
$$\widehat{zz} = \frac{\partial}{\partial z}\left\{(2-\sigma)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right\}, \quad \widehat{\phi\phi} = \frac{\partial}{\partial z}\left\{\sigma\nabla^2\chi - \frac{1}{r}\frac{\partial\chi}{\partial r}\right\}. \tag{7}$$

Here and in what follows $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ denotes the Laplace operator, the subjects of operation being independent of the circumferential coordinate $\phi$. The quantities $E = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$ and $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ are Young's modulus and Poisson's ratio, respectively.

The function $\chi$ satisfies the biharmonic equation

$$\nabla^2\nabla^2\chi = 0\,. \tag{8}$$

Since then Love's representation of the general solution was widely used by Timoshenko [7–9]. It is interesting to note that much later Filon also used that representation when considering [20, Section 7.04], the problem of elastic equilibrium of an infinite cylinder under the action of concentrated shearing forces at its surface.

### 3.2. Filon's solution

Filon [1] presented the solutions of Equations in terms of combinations of trigonometric and Bessel functions and constructed the representations for the displacement components as follows:

$$u = u_0\, r - \sum_{n=0}^{\infty}\left[A_1^{(n)} I_1(\kappa_n r) + C^{(n)} r I_0(\kappa_n r)\right]\frac{\cos\kappa_n z}{\kappa_n},$$
$$w = w_0\, z + \sum_{n=0}^{\infty}\left[A_2^{(n)} I_0(\kappa_n r) + C^{(n)} r I_1(\kappa_n r)\right]\frac{\sin\kappa_n z}{\kappa_n}, \tag{9}$$

where $\kappa_n = \frac{(2n+1)\pi}{2c}$. Here

$$u_0 = \frac{\lambda+2\mu}{2\mu(3\lambda+2\mu)}\, p(c)\,, \quad w_0 = -\frac{\lambda}{\mu(3\lambda+2\mu)}\, p(c)\,,$$

and the coefficients $A_1^{(n)}$, $A_2^{(n)}$, and $C^{(n)}$ are expressed via Fourier coefficients of expansions of the functions $p(z) - p(c)$ and $s(z)$ on complete systems $\cos\kappa_n z$ and $\sin\kappa_n z$, respectively. The corresponding expressions for the stresses $\widehat{rr}$, $\widehat{zz}$, and $\widehat{rz}$ are recorded in [1, p. 157].

For the particular case of discontinuous pure shearing loading (4) Filon gave explicit expressions for the displacements and stresses, and performed extensive numerical calculations with these Fourier series. He presented the results in several diagrams and tables that show the distributions of displacements and stresses in the cylinder. Some of these data are reproduced below in Figures 1*a*–3*a* and Tables 3–4 (in braces).

The solution (9) provides zero normal stress $\widehat{zz}$ at the plane ends of the cylinder, but it also gives a system of finite shear stress $\widehat{rz}$, which is, however, self-equilibrating. The shear is zero at the center and at the circumference, and its greatest value does not exceed about 0·25S. Filon stated that the effects of this system, at some distance inside the cylinder, will therefore (according to the Saint-Venant principle) be small compared with the effects of the large and unbalanced lateral distribution of shear.

### 3.3. General solution of the boundary-value problem

We employ the method of superposition to construct the solution of the boundary-value problem (1)–(3). The idea of this method consists of using the sum of two ordinary series on the complete systems of trigonometric and Bessel functions in $z$ and $r$ coordinates, respectively. These series both satisfy identically the governing equations inside the cylindrical domain and have sufficient functional arbitrariness for fulfilling the two boundary conditions on the curved surface and the two flat ends. Because of the interdependency, the expression for a coefficient of a term in one series will depend on all coefficients of the other series and *vice versa*. Therefore, the final solution requires the solution of the infinite system of linear algebraic equations giving the relations between the coefficients and loading forces.

To construct the general solution of Equations (1) adjusted with the boundary conditions (3) let us choose the biharmonic function $\chi$ in the form

$$\begin{aligned}\chi &= B_0 z^3 + \sum_{j=1}^{\infty}\Big(A_j \frac{\sinh \lambda_j z}{\sinh \lambda_j c} + B_j z \frac{\cosh \lambda_j z}{\sinh \lambda_j c}\Big)\frac{J_0(\lambda_j r)}{\lambda_j^2} \\ &+ D_0 r^2 z + \sum_{n=1}^{\infty}\left[C_n \frac{I_0(k_n r)}{I_1(k_n a)} + D_n r \frac{I_1(k_n r)}{I_1(k_n a)}\right]\frac{\sin k_n z}{k_n^2}\,,\end{aligned} \tag{10}$$

where $k_n = \dfrac{n\pi}{c}$, and $\lambda_j$ is a non-zero root of the equation $J_1(\lambda_j a) = 0$. Real constants $A_n, B_n, C_j, D_j$ are yet arbitrary; they are to be determined from the boundary conditions.

The stresses corresponding to (10) are given by Equations (7):

$$\begin{aligned}\widehat{rr} &= 6\sigma B_0 + (4\sigma - 2)D_0 \\ &+ \sum_{j=1}^{\infty}\left\{A_j\lambda_j \frac{\cosh \lambda_j z}{\sinh \lambda_j c} + B_j\left[(1+2\sigma)\frac{\cosh \lambda_j z}{\sinh \lambda_j c} + \lambda_j z\,\frac{\sinh \lambda_j z}{\sinh \lambda_j c}\right]\right\} J_0(\lambda_j r) \\ &- \sum_{j=1}^{\infty}\left[A_j\lambda_j \frac{\cosh \lambda_j z}{\sinh \lambda_j c} + B_j\left(\frac{\cosh \lambda_j z}{\sinh \lambda_j c} + \lambda_j z\,\frac{\sinh \lambda_j z}{\sinh \lambda_j c}\right)\right]\frac{J_1(\lambda_j r)}{\lambda_j r} \\ &- \sum_{n=1}^{\infty}\left\{\frac{C_n}{I_1(k_n a)}\,\frac{dI_1(k_n r)}{dr} + D_n\left[k_n r\,\frac{I_1(k_n r)}{I_1(k_n a)} + (1-2\sigma)\frac{I_0(k_n r)}{I_1(k_n a)}\right]\right\}\cos k_n z\,,\end{aligned} \tag{11}$$

$$\begin{aligned}\widehat{zz} &= (6-6\sigma)B_0 + (8-4\sigma)D_0 \\ &- \sum_{j=1}^{\infty}\left\{A_j\lambda_j \frac{\cosh \lambda_j z}{\sinh \lambda_j c} + B_j\left[(2\sigma-1)\frac{\cosh \lambda_j z}{\sinh \lambda_j c} + \lambda_j z\,\frac{\sinh \lambda_j z}{\sinh \lambda_j c}\right]\right\} J_0(\lambda_j r) \\ &+ \sum_{n=1}^{\infty}\left\{C_n k_n \frac{I_0(k_n r)}{I_1(k_n a)} + D_n\left[(4-2\sigma)\frac{I_0(k_n r)}{I_1(k_n a)} + k_n r\,\frac{I_1(k_n r)}{I_1(k_n a)}\right]\right\}\cos k_n z\,,\end{aligned} \tag{12}$$

$$\widehat{rz} = \sum_{j=1}^{\infty} \left\{ A_j \lambda_j \frac{\sinh \lambda_j z}{\sinh \lambda_j c} + B_j \left[ 2\sigma \frac{\sinh \lambda_j z}{\sinh \lambda_j c} + \lambda_j z \, \frac{\sinh \lambda_j z}{\sinh \lambda_j c} \right] \right\} J_1(\lambda_j r)$$

$$+ \sum_{n=1}^{\infty} \left\{ C_n k_n \frac{I_1(k_n r)}{I_1(k_n a)} + D_n \left[ (2-2\sigma) \frac{I_1(k_n r)}{I_1(k_n a)} + k_n r \, \frac{I_0(k_n r)}{I_1(k_n a)} \right] \right\} \sin k_n z \, . \tag{13}$$

The structure of these expressions is obvious: the Dini and Fourier-Bessel series on the complete systems $\{1, J_0(\lambda_j r)\}$ and $J_1(\lambda_j r)$, respectively, with two sets of constants $A_j$ and $B_j$ are capable of representing *any* assigned distribution of $\widehat{zz}$ and $\widehat{zr}$ at $z = \pm c$. Similarly, the Fourier series on the complete trigonometric systems $\{1, \cos k_n z\}$ and $\sin k_n z$ with two sets of arbitrary constants $C_n$ and $D_n$ can represent any assigned distribution of $\widehat{rr}$ and $\widehat{rz}$ at $r = a$. Therefore, from the very beginning the expressions for $\widehat{rr}$, $\widehat{zz}$ and $\widehat{rz}$ appear to be capable to represent any stress distribution for which the boundary conditions (3) have to be satisfied.

To proceed further with this solution, let assume that the surface shearing load $s(z)$ is represented by the Fourier series

$$s(z) = \sum_{n=1}^{\infty} (-1)^n \, s_n \sin k_n z \, , \qquad s_n = \frac{(-1)^n}{c} \int_{-c}^{c} s(z) \sin k_n z \, \mathrm{d}z \, . \tag{14}$$

Then the boundary conditions for $\widehat{rz}$ everywhere on the surface of the cylinder lead to the relations

$$\begin{aligned} A_j \lambda_j &= -B_j \, (2\sigma + \lambda_j c \coth \lambda_j c) \, , \\ C_n k_n &= -D_n \left[ 2 - 2\sigma + k_n a \frac{I_0(k_n a)}{I_1(k_n a)} \right] + (-1)^n s_n \, . \end{aligned} \tag{15}$$

With these relations the boundary conditions (3) for normal stresses at the surface of the cylinder provide two functional equations containing two sequences of constants $B_j$ and $D_n$:

$$p(z) = 6\sigma B_0 + (4\sigma - 2) D_0$$

$$+ \sum_{j=1}^{\infty} B_j \left[ (1 - \lambda_j c \coth \lambda_j c) \frac{\cosh \lambda_j z}{\sinh \lambda_j c} + \lambda_j z \, \frac{\sinh \lambda_j z}{\sinh \lambda_j c} \right] J_0(\lambda_j a)$$

$$+ \sum_{n=1}^{\infty} \left\{ D_n \left[ k_n a \, \left( d_n^2 - 1 \right) - \frac{2 - 2\sigma}{k_n a} \right] - (-1)^n s_n \left( d_n - \frac{1}{k_n a} \right) \right\} \cos k_n z \, , \tag{16}$$

$$0 = (6 - 6\sigma) B_0 + (8 - 4\sigma) D_0 + \sum_{j=1}^{\infty} B_j \left( \coth \lambda_j c + \frac{\lambda_j c}{\sinh^2 \lambda_j c} \right) J_0(\lambda_j r)$$

$$+ \sum_{n=1}^{\infty} \left\{ (-1)^n D_n \left[ k_n r \, \frac{I_1(k_n r)}{I_1(k_n a)} + (2 - k_n a d_n) \, \frac{I_0(k_n r)}{I_1(k_n a)} \right] + s_n \frac{I_0(k_n r)}{I_1(k_n a)} \right\} \, , \tag{17}$$

with the notation

$$d_n = \frac{I_0(k_n a)}{I_1(k_n a)} \, .$$

Multiplying both sides of Equation (16) by $\cos k_n z\, \mathrm{d}z$ and integrating over the interval $(-c, c)$, and multiplying the both sides of the Equation (17) by $r J_0(\lambda_j r)\, \mathrm{d}r$ and integrating over the interval $(0, a)$ or, equivalently, by expanding the normal stresses $\widehat{rr}$ at $r = a$ into the Fourier series and the normal stresses $\widehat{zz}$ at $z = c$ into the Dini series by means of expansions (A1), (A4), (A5), after some computations we obtain two equations for defining $B_0$ and $D_0$

$$6\sigma B_0 + (4\sigma - 2) D_0 = p_0\,, \quad (6 - 6\sigma) B_0 + (8 - 4\sigma) D_0 = g_0\,, \tag{18}$$

and an infinite system of linear algebraic equations:

$$\left.\begin{aligned} X_n P_n - \sum_{j=1}^{\infty} Y_j \frac{4k_n^2}{\left(\lambda_j^2 + k_n^2\right)^2} &= f_n\,, \qquad n = 1, 2, \ldots \\ Y_j \Delta_j - \sum_{n=1}^{\infty} X_n \frac{4\lambda_j^2}{\left(k_n^2 + \lambda_j^2\right)^2} &= g_j\,, \qquad j = 1, 2, \ldots \end{aligned}\right\}\,, \tag{19}$$

where new unknown coefficients $X_n$ and $Y_j$ appear instead of $D_n$ and $B_j$:

$$X_n = (-1)^n D_n \frac{k_n}{a} - \frac{s_n k_n}{a}\,, \quad Y_j = -B_j \frac{\lambda_j}{c} J_0(\lambda_j a) \tag{20}$$

and the notations

$$P_n = a^2 \left(d_n^2 - 1\right) - \frac{2 - 2\sigma}{k_n^2}\,, \quad \Delta_j = \frac{c}{\lambda_j} \left(\coth \lambda_j c + \frac{\lambda_j c}{\sinh^2 \lambda_j c}\right) \tag{21}$$

are introduced. The right-hand sides in Equations (18) and (19) are defined as follows

$$f_n = p_n + s_n \left[k_n a \left(1 - d_n^2\right) + d_n + \frac{1 - 2\sigma}{k_n a}\right]\,, \tag{22}$$

$$g_0 = -\frac{2}{a} \sum_{n=1}^{\infty} \frac{s_n}{k_n}\,, \qquad g_j = \frac{4\lambda_j^2}{a} \sum_{n=1}^{\infty} \frac{s_n k_n}{\left(k_n^2 + \lambda_j^2\right)^2} + \frac{2}{a} \sum_{n=1}^{\infty} \frac{s_n k_n}{k_n^2 + \lambda_j^2}\,. \tag{23}$$

Here the sequence $p_0$, $(-1)^n p_n$ defined as

$$p_0 = \frac{1}{2c} \int_{-c}^{c} p(z)\, \mathrm{d}z\,, \quad (-1)^n p_n = \frac{1}{c} \int_{-c}^{c} p(z) \cos k_n z\, \mathrm{d}z \tag{24}$$

represents the Fourier coefficients of the normal loading $p(z)$ with

$$p(z) = p_0 + \sum_{n=1}^{\infty} (-1)^n p_n \cos k_n z\,. \tag{25}$$

Next, the definition (14) of $s_n$ and expansions (A2), (A3) permit one to represent the expressions for $g_0$ and $g_j$ in the form of rather simple integrals:

$$\begin{aligned} g_0 &= \frac{1}{ac} \int_{-c}^{c} s(z) z\, \mathrm{d}z\,, \\ g_j &= \frac{1}{a} \int_{-c}^{c} s(z) \left[\lambda_j z \frac{\cosh \lambda_j z}{\sinh \lambda_j c} - (1 + \lambda_j c \coth \lambda_j c) \frac{\sinh \lambda_j z}{\sinh \lambda_j c}\right] \mathrm{d}z\,. \end{aligned} \tag{26}$$

The system consisting of the two Equations (18) for the determination of $B_0$ and $D_0$ corresponds to the elementary solution with uniform distribution of normal stresses throughout the finite cylinder. It is interesting to note that the values $B_0$ and $D_0$ depend not only upon the non-equilibrating part of the normal loading at the cylindrical surface, but also upon of the distribution of shearing loadings.

### 3.4. Analysis of the infinite system

An obvious way to obtain a solution of the infinite system (19) consists of a simple reduction to a finite system. In other words, this is achieved by regarding the number of unknowns and equations as finite by setting

$$X_n = 0, \quad (n > N) \qquad Y_j = 0, \quad (j > J)\,, \tag{27}$$

solving the finite system of $N + J$ equations to obtain approximate values for a finite number of first unknowns $X_n$ and $Y_j$, and then finding the limits towards which the values of unknowns converge as the numbers $N$ and $J$ of equations involved increase. The justification of such an approach is based on the notion of regular infinite systems.

By means of sums (A7) and (A9) we have the the equalities

$$\begin{aligned} &\frac{1}{P_n}\sum_{j=1}^{\infty}\frac{4k_n^2}{\left(\lambda_j^2+k_n^2\right)^2} = 1-\psi_n\,, \qquad \psi_n = \frac{2+2\sigma)}{k_n^2 P_n} \\ &\frac{1}{\Delta_j}\sum_{n=1}^{\infty}\frac{4\lambda_j^2}{\left(k_n^2+\lambda_j^2\right)^2} = 1-\phi_j\,, \qquad \phi_j = \frac{2}{\lambda_j^2\Delta_j}\,. \end{aligned} \tag{28}$$

Obviously $\Delta_j > 0$ and it can be shown, following [10], that $P_n > 0$. Next, the sequences $\psi_n$ and $\phi_j$ are positive definite and none of their terms becomes equal to unity for all values of $n$ and $j$. This means that the infinite system (19) is *regular*. According to the general theory of infinite systems [21], the regular infinite system (19) has a *unique bounded* solution, if its right-hand-side terms $\dfrac{f_n}{P_n}$ and $\dfrac{g_j}{\Delta_j}$ decrease at least at the same rate (or faster) as the sequences $\psi_n$ and $\phi_j$ when $n \to \infty$ and $j \to \infty$, respectively.

It can be readily shown by means of Equation (A12) that

$$P_n = \frac{a}{k_n} - \frac{1-2\sigma}{k_n^2} + O\left(\frac{1}{n^3}\right), \qquad \psi_n = O\left(\frac{1}{n}\right), \quad n \to \infty\,, \tag{29}$$

$$\Delta_j = \frac{c}{\lambda_j} + O(\exp(-j))\,, \qquad \phi_j = O\left(\frac{1}{j}\right), \quad j \to \infty\,. \tag{30}$$

If the functions $p(z)$ and $s(z)$ are continuous functions with continuous second derivatives, one obtains by means of the usual procedure of integration by parts in (24) and (26),

$$f_n = \frac{2p'(c)}{ck_n^2} + O\left(\frac{1}{n^3}\right), \quad n \to \infty\,, \qquad g_j = \frac{2s'(c)}{a\lambda_j^2} + O\left(\frac{1}{j^3}\right), \quad j \to \infty\,. \tag{31}$$

For Filon's example (4) of discontinuous tangent loading $s(z)$ the sequence $s_n$ is

$$s_n = (-1)^n\frac{2S}{ck_n}(\cos k_n c_1 - \cos k_n c_2)\,, \tag{32}$$

and the free terms $f_n$ and $g_j$ due to Equation (A12) have asymptotic behaviour

$$f_n = O\left(\frac{1}{n^2}\right), \quad n \to \infty, \qquad g_j = O(\exp(-j)), \quad j \to \infty. \tag{33}$$

Comparing Equations (31) and (33) with Equations (29) and (30) we conclude that the condition for existence of the unique bounded solution is fulfilled for both cases of loading.

The method of simple reduction is a traditional approach for the wide range of boundary-value problems in engineering mathematics. The solutions of these problems are usually chosen in such a way that terms in the Fourier series like Equations (11)–(13) decrease exponentially when proceeding from the boundary into the domain. Therefore, any small variations in the coefficients $X_n$ and $Y_j$ would not change considerably the main field inside. It is important to note that the governing equations inside a domain are satisfied identically for *any* values of the coefficients $X_n$ and $Y_j$, and the accuracy of satisfying the prescribed boundary conditions is the single criterion for the quality of the solution. In the majority of publications this principal question has been pointed out only scarcely, though; see [18] for an exellent discussion of all the details.

To obtain the correct values of the normal stresses near the rim we need to know the asymptotic behaviour of the unknowns $X_n$ and $Y_j$ for $n \to \infty$ and $j \to \infty$, respectively. Based upon Koialovich's [22] theory of limitants, Grinchenko [15, 16] established that the asymptotic behaviour of the unknowns is

$$\lim_{n\to\infty} X_n = \lim_{j\to\infty} Y_j = G \tag{34}$$

with some mutual constant $G$ which is, in general, not equal to zero. He suggested the improved reduction approach by setting in the first $N$ and $J$ equations in (19)

$$X_n = X_N, \quad n > N, \qquad Y_j = Y_J, \quad j > J, \tag{35}$$

for sufficiently large $N$ and $J$, which replaces the infinite system by a finite system of $N + J$ equations. This method of improved reduction allows us to considerably increase the accuracy in finding *all* unknowns based on the solutions of the first equations.

We propose another effective approach (which is closely connected with the behaviour of the stress field near the rim of the cylinder, as it will be shown below) of finding the value $G$ which runs as follows. Let us put

$$X_n = G + x_n, \qquad Y_j = G + y_j, \tag{36}$$

with the asymptotic behaviour of the new unknowns $x_n$ and $y_j$

$$x_n = o(1), \quad n \to \infty, \qquad y_j = o(1), \quad j \to \infty. \tag{37}$$

Then by using the values of sums (A7) and (A9) and notations (21), one obtains

$$\left.\begin{aligned} &x_n P_n - \sum_{j=1}^{\infty} y_j \frac{4k_n^2}{\left(\lambda_j^2 + k_n^2\right)^2} + G\,\frac{2+2\sigma}{k_n^2} = f_n, \quad n = 1, 2, \ldots \\ &y_j \Delta_j - \sum_{n=1}^{\infty} x_n \frac{4\lambda_j^2}{\left(k_n^2 + \lambda_j^2\right)^2} + G\,\frac{2}{\lambda_j^2} = g_j, \quad j = 1, 2, \ldots \end{aligned}\right\} . \tag{38}$$

Next, let us sum the first set of Equations (38) with respect to $n$, and the second set with respect to $j$ and then change the order of summing in the double series. These operations are justified by the asymptotic behaviour of $x_n$ and $y_j$, $P_n$ and $\Delta_j$, and $f_n$ and $g_j$ as given by Equations (37), (29–31). By means of the sums (A8), (A10), (A11) one obtains two equations

$$\sum_{n=1}^{\infty} x_n\, P_n - \sum_{j=1}^{\infty} y_j\, \frac{c}{\lambda_j}\left(\coth\lambda_j c - \frac{\lambda_j c}{\sinh^2\lambda_j c}\right) + G\,\frac{(1+\sigma)c^2}{3} = \sum_{n=1}^{\infty} f_n\,,$$

$$\sum_{j=1}^{\infty} y_j\,\Delta_j - \sum_{n=1}^{\infty} x_n\left[a^2\left(1-d_n^2\right) + \frac{2ad_n}{k_n}\right] + G\,\frac{a^2}{4} = \sum_{j=1}^{\infty} g_j\,.$$

Finally, adding these two equations and taking into account Equations (25) when $z = c$ and (23) for $g_j$ one obtains

$$G\left[\frac{(1+\sigma)c^2}{3} + \frac{a^2}{4}\right] = p(c) - p_0 + \sum_{n=1}^{\infty} s_n\left[2k_n a\left(1-d_n^2\right) + 4d_n - \frac{1+2\sigma}{2k_n a}\right]$$

$$-\sum_{n=1}^{\infty} x_n\left[2a^2\left(1-d_n^2\right) + \frac{4ad_n}{k_n} + \frac{2-2\sigma}{k_n^2}\right] - \sum_{j=1}^{\infty} y_j\frac{2c^2}{\sinh^2\lambda_j c}\,. \tag{39}$$

In this equation the multiples of $x_n$ and $y_j$ in the right-hand-side sums decrease rapidly, especially in the sum with respect to $j$. Therefore, the values of the first few terms of $x_n$ and $y_j$ can provide an accurate value of $G$ which does not essentially change when the numbers $N$ and $J$ in the finite system are increased.

Thus, the finite number of the coefficients $x_n$, $y_j$, and $G$ can be found when solving the *finite* system corresponding to (38), leaving in it only the first $N$ and $J$ unknowns $x_n$ and $y_j$, respectively, and the additional Equation (39). It is important to note that the solution of this finite system provides, according to (36), knowledge about *all* coefficients $X_n$ and $Y_j$.

### 3.5. Stress field in the cylinder

By means of the sequences of coefficients $x_n$ $(n = 1, \ldots, N)$, $y_j$ $(j = 1, \ldots, J)$, $G$, and $s_n$ $(n = 1, 2, \ldots)$, the biharmonic Love function can be written as

$$\chi = B_0 z^3 + D_0 r^2 z + \chi_s + G\chi_G$$

$$+c\sum_{j=1}^{J} y_j\, Z(z)\,\frac{J_0(\lambda_j r)}{\lambda_j^4\, J_0(\lambda_j a)} + a\sum_{n=1}^{N}(-1)^n\,\frac{x_n}{k_n^4}\,R(r)\,\sin k_n z\,, \tag{40}$$

where

$$\chi_s = \sum_{n=1}^{\infty}(-1)^n\,\frac{s_n}{k_n^3}\left[k_n r\,\frac{I_1(k_n r)}{I_1(k_n a)} - (1-2\sigma+k_n a d_n)\frac{I_0(k_n r)}{I_1(k_n a)}\right]\sin k_n z\,, \tag{41}$$

$$\chi_G = c\sum_{j=1}^{\infty} Z(z)\,\frac{J_0(\lambda_j r)}{\lambda_j^4\, J_0(\lambda_j a)} + a\sum_{n=1}^{\infty}\frac{(-1)^n}{k_n^4}\,R(r)\,\sin k_n z\,. \tag{42}$$

and

$$Z(z) = (2\sigma + \lambda_j c \coth \lambda_j c) \frac{\sinh \lambda_j z}{\sinh \lambda_j c} - \lambda_j z \frac{\cosh \lambda_j z}{\sinh \lambda_j c},$$

$$R(r) = k_n r \frac{I_1(k_n r)}{I_1(k_n a)} - (2 - 2\sigma + k_n a d_n) \frac{I_0(k_n r)}{I_1(k_n a)}.$$

By means of series expansions (A1)–(A6) and after some transformations, we obtain

$$\chi_G = (4\sigma - 8) S_5(z) + \sum_{j=1}^{\infty} \left[ \frac{4\sigma - 4}{\lambda_j^2} S_3(z) - \frac{4\sigma}{\lambda_j^4} S_1(z) \right] \frac{J_0(\lambda_j r)}{J_0(\lambda_j a)} \tag{43}$$

where the sums are defined as follows:

$$S_1(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{k_n} \sin k_n z = -\frac{z}{2}, \quad S_3(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{k_n^3} \sin k_n z = \frac{z^3 - zc^2}{12},$$

$$S_5(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{k_n^5} \sin k_n z = \frac{3z^5 + 10z^3c^2 - zc^4}{720},$$

and, after summation with respect to $r$

$$\chi_G = (4\sigma - 8) S_5(z) + (4\sigma - 4) S_3(z) R_2(r) - 4\sigma S_1(z) R_4(r), \tag{44}$$

where

$$R_2(r) = \sum_{j=1}^{\infty} \frac{J_0(\lambda_j r)}{\lambda_j^2 J_0(\lambda_j a)} = \frac{r^2}{4} - \frac{a^2}{8},$$

$$R_4(r) = \sum_{j=1}^{\infty} \frac{J_0(\lambda_j r)}{\lambda_j^4 J_0(\lambda_j a)} = \frac{a^4}{192} + \frac{r^2 a^2}{32} - \frac{r^4}{64},$$

By means of (7) and (40) we can straightforwardly express the stresses in the whole cylinder, including its boundary with the rims. The terms corresponding to $G$ are

$$\widehat{rr}_G = G \left[ \frac{1+\sigma}{2} \left( z^2 - \frac{c^2}{3} \right) - \sigma \frac{r^2 - a^2}{8} \right], \quad \widehat{zz}_G = G \left( \frac{a^2}{4} - \frac{r^2}{2} \right),$$

$$\widehat{rz}_G = 0, \qquad \widehat{\phi\phi}_G = G \left[ \sigma \left( z^2 - \frac{c^2}{3} \right) - \sigma \left( \frac{r^2}{2} - \frac{a^2}{4} \right) - 1 \right]. \tag{45}$$

It is important that there is a *finite* input of terms with factor $G$ into the values of the normal stresses $\widehat{rr}$, $\widehat{zz}$, $\widehat{\phi\phi}$ on the rim $r = a$, $z = c$. This proves that the simple reduction method (27) of solving the infinite system (19) produces errors in the determination of the normal stresses $\widehat{rr}$, $\widehat{zz}$, $\widehat{\phi\phi}$ near the rim of the cylinder due to the nonuniform convergence of the series with functions like $\dfrac{I_0(k_n r)}{I_1(k_n a)}$ and $\dfrac{\cosh \lambda_j z}{\sinh \lambda_j c}$ near $r = a$ and $z = c$, respectively. These errors can not be decreased by increasing indefinitely the numbers $N$ and $J$ in the reduced finite system. This circumstance was first pointed out by Grinchenko [18].

The convergence of the infinite series for that part of the stresses which depend upon coefficients $s_n$ can be accelerated by means of the semi-convergent expansions of the modified Bessel functions when their arguments are fairly large

$$\frac{I_0(kr)}{I_1(ka)} = \sqrt{\frac{a}{r}}\,\mathrm{e}^{-k(a-r)}\left(1 + \frac{1}{8kr} + \frac{3}{8ka} + O(k^{-2})\right),$$

$$\frac{I_1(kr)}{I_1(ka)} = \sqrt{\frac{a}{r}}\,\mathrm{e}^{-k(a-r)}\left(1 - \frac{3}{8kr} + \frac{3}{8ka} + O(k^{-2})\right)$$

when $k \to \infty$ and $r$ approaches $a$. Doing so we obtain

$$\begin{aligned}(\widehat{rr})_s &= \sum_{n=1}^{\infty}(-1)^n s_n \left\{\left[k_n a d_n \frac{I_0(k_n r)}{I_1(k_n a)} - k_n r\frac{I_1(k_n r)}{I_1(k_n a)} - d_n \frac{a I_1(k_n r)}{r I_1(k_n a)}\right.\right.\\ &\quad \left.-\frac{(1-2\sigma)I_1(k_n r)}{k_n r I_1(k_n a)}\right] - \left.\left[k_n(a-r) + \frac{5}{4} - \frac{7a}{8r} - \frac{3r}{8a}\right]\sqrt{\frac{a}{r}}\,\mathrm{e}^{-k_n(a-r)}\right\}\cos k_n z \\ &\quad +(a-r)\sqrt{\frac{a}{r}}\,S^{(1)}(r,z) + \left(\frac{5}{4} - \frac{7a}{8r} - \frac{3r}{8a}\right)\sqrt{\frac{a}{r}}\,S^{(0)}(r,z)\,,\end{aligned} \tag{46}$$

$$\begin{aligned}(\widehat{zz})_s &= \sum_{n=1}^{\infty}(-1)^n s_n \left\{\left[k_n r\frac{I_1(k_n r)}{I_1(k_n a)} + (3 - k_n a d_n)\frac{I_0(k_n r)}{I_1(k_n a)}\right]\right.\\ &\quad \left.-\left[k_n(r-a) + \frac{7}{4} - \frac{a}{8r} + \frac{3r}{8a}\right]\sqrt{\frac{a}{r}}\,\mathrm{e}^{-k_n(a-r)}\right\}\cos k_n z \\ &\quad +(r-a)\sqrt{\frac{a}{r}}\,S^{(1)}(r,z) + \left(\frac{7}{4} - \frac{a}{8r} + \frac{3r}{8a}\right)\sqrt{\frac{a}{r}}\,S^{(0)}(r,z)\,,\end{aligned} \tag{47}$$

$$\begin{aligned}(\widehat{rz})_s &= \sum_{n=1}^{\infty}(-1)^n s_n \left\{\left[k_n r\frac{I_0(k_n r)}{I_1(k_n a)} + (1 - k_n a d_n)\frac{I_1(k_n r)}{I_1(k_n a)}\right]\right.\\ &\quad \left.-\left[k_n(r-a) + \frac{1}{4} + \frac{3a}{8r} + \frac{3r}{8a}\right]\sqrt{\frac{a}{r}}\,\mathrm{e}^{-k_n(a-r)}\right\}\sin k_n z \\ &\quad +(r-a)\sqrt{\frac{a}{r}}\,T^{(1)}(r,z) + \left(\frac{1}{4} + \frac{3a}{8r} + \frac{3r}{8a}\right)\sqrt{\frac{a}{r}}\,T^{(0)}(r,z)\,,\end{aligned} \tag{48}$$

where the notations

$$\begin{aligned}S^{(1)}(r,z) &= \sum_{n=1}^{\infty}(-1)^n s_n k_n \mathrm{e}^{-k_n(a-r)}\cos k_n z\,,\\ S^{(0)}(r,z) &= \sum_{n=1}^{\infty}(-1)^n s_n \mathrm{e}^{-k_n(a-r)}\cos k_n z\,,\end{aligned} \tag{49}$$

and

$$\begin{aligned}T^{(1)}(r,z) &= \sum_{n=1}^{\infty}(-1)^n s_n k_n \mathrm{e}^{-k_n(a-r)}\sin k_n z\,,\\ T^{(0)}(r,z) &= \sum_{n=1}^{\infty}(-1)^n s_n \mathrm{e}^{-k_n(a-r)}\sin k_n z\,.\end{aligned} \tag{50}$$

are introduced.

For the concrete values of $s_n$ in Filon's example given by Equation (32) we obtain

$$
\begin{aligned}
S^{(1)}(r,z) \;&= \frac{S}{c}\,[R_0(r,c_1+z)+R_0(r,c_1-z)\\
&\quad -\,R_0(r,c_2+z)-R_0(r,c_2-z)]\ ,\\
S^{(0)}(r,z) \;&= \frac{S}{c}\left[R_{-1}(r,c_1+z)+R_{-1}(r,c_1-z)\right.\\
&\quad \left.-\,R_{-1}(r,c_2+z)-R_{-1}(r,c_2-z)\right]\ ,
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
T^{(1)}(r,z) \;&= \frac{S}{c}\,[T_0(r,c_1+z)-T_0(r,c_1-z)\\
&\quad -\,T_0(r,c_2+z)+T_0(r,c_2-z)]\ ,\\
T^{(0)}(r,z) \;&= \frac{S}{c}\left[T_{-1}(r,c_1+z)-T_{-1}(r,c_1-z)\right.\\
&\quad \left.-\,T_{-1}(r,c_2+z)+T_{-1}(r,c_2-z)\right]\ ,
\end{aligned}
\tag{52}
$$

where according to Equations (1.75)–(1.78) in [23],

$$
R_0(r,\zeta) = \sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-k_n\rho}\cos k_n\zeta \;=\; \frac{1}{2}\left[\frac{\sinh\frac{\pi\rho}{c}}{\cosh\frac{\pi\rho}{c}-\cos\frac{\pi\zeta}{c}}-1\right],
\tag{53}
$$

$$
R_{-1}(r,\zeta) = \sum_{n=1}^{\infty}\frac{(-1)^n}{k_n}\,\mathrm{e}^{-k_n\rho}\cos k_n\zeta \;=\; \frac{1}{2}\rho-\frac{c}{2\pi}\ln\left[2\cosh\frac{\pi\rho}{c}-2\cos\frac{\pi\zeta}{c}\right],
$$

and

$$
T_0(r,\zeta) = \sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-k_n\rho}\sin k_n\zeta \;=\; \frac{\sin\frac{\pi\zeta}{c}}{2\cosh\frac{\pi\rho}{c}-2\cos\frac{\pi\zeta}{c}}\ ,
\tag{54}
$$

$$
T_{-1}(r,\zeta) = \sum_{n=1}^{\infty}\frac{(-1)^n}{k_n}\,\mathrm{e}^{-k_n\rho}\sin k_n\zeta \;=\; \frac{c}{\pi}\arctan\frac{\sin\frac{\pi\zeta}{c}}{\exp\frac{\pi\rho}{c}-\cos\frac{\pi\zeta}{c}}\ ,
$$

with $\rho = a - r$.

From these expressions one may conclude that the shear stress $\widehat{rz}$ is continuous inside the cylinder, and at the surface $r = a$ it provides the representation for the discontinuous function $s(z)$. It is easy to verify that the functions $S^{(1)}(r,z)$ and $S^{(0)}(r,z)$ tend to infinity near the points $(a, \pm c_1)$ and $(a, \pm c_2)$. However, due to multiple $a - r$ the radial stress $\widehat{rr}$ remains finite and continuous everywhere in the cylinder including these points. On the other hand, the axial stress $\widehat{zz}$ and the circumferential stress $\widehat{\phi\phi}$ (not written here) really tend to $\pm\infty$ as we approach the points of discontinuity of applied shearing loads. In practice this means that, as the transition from the shearing stressed to unstressed surface becomes more abrupt, the normal stresses in the neighbourhood become dangerously large.

*Table 1.* Radial stress $\widehat{rr}/p_0$ at the boundary $r = a$ of the cylinder for various numbers $(N, J)$ of unknowns found by the improved reduction and simple reduction, the infinite systems (38), (39) and (19), respectively

| | Exact | Improved reduction | | Simple reduction | | | |
|---|---|---|---|---|---|---|---|
| $z/c$ | $p(z)/p_0$ | (1,1) | (5,5) | (10,10) | (20,20) | (30,30) | (40,40) |
| 0·0 | 1·500 | 1·521 | 1·500 | 1·485 | 1·492 | 1·495 | 1·499 |
| 0·2 | 1·440 | 1·449 | 1·440 | 1·426 | 1·433 | 1·435 | 1·438 |
| 0·4 | 1·260 | 1·245 | 1·260 | 1·246 | 1·253 | 1·255 | 1·256 |
| 0·6 | 0·960 | 0·934 | 0·960 | 0·947 | 0·953 | 0·955 | 0·956 |
| 0·8 | 0·540 | 0·542 | 0·538 | 0·541 | 0·535 | 0·536 | 0·557 |
| 1·0 | 0·000 | 0·039 | 0·001 | 0·467 | 0·469 | 0·470 | 0·470 |

Now, all the series for the stresses are rapidly convergent everywhere and for practical calculations only a small number of terms is sufficient to provide high accuracy. It is worth noting that by calculating the difference $\widehat{rr} - \widehat{zz}$ at the rim $r = a$, $z = c$, which according to the boundary conditions (3) is equal to $p(c)$, we arrive at Equation (39).

## 4. Results and discussion

The advantages of the method of superposition, namely relatively little calculation compared to other approaches, and high accuracy in the determination of the stress field everywhere, including the boundary near circumference of the cylinder, become more evident when two typical examples of loading of the cylinder are considered. For both examples we choose Filon's parameters of the cylinder, $2c = \pi a$, $\sigma = 0{\cdot}25$.

Let us consider first a parabolically distributed normal loading

$$p(z) = \frac{3}{2}\, p_0 \left(1 - \frac{z^2}{c^2}\right), \tag{55}$$

with total intensity $p_0 =$ const. Shearing load is absent, that is $s(z) = 0$.

In Table 1 we present a comparison between two approaches to the solution of the infinite system (19) – improved reduction and simple reduction, respectively – when satisfying the prescribed boundary conditions for $\widehat{rr}$ at $r = a$. It can be seen that already five leading terms in each series along with terms given by Equation (18) in the improved reduction algorithm provide an excellent agreement with the prescribed parabolic distribution of the normal load. At the same time, even forty terms in the simple reduction approach do not provide an accurate satisfaction of the boundary condition at the rim. This inaccuracy can not be decreased by increasing indefinitely the numbers $N$ and $J$ in the reduced finite system (19).

Inside the cylinder for $0 \le r \le 0{\cdot}8a$, $|z| \le 0{\cdot}8c$, the values of the normal stresses defined by both approaches do not differ significantly, see Table 2. Here also the developed approach requires considerably fewer terms in the Fourier series. Moreover, using only the term with $G$ in (40), that is, putting $N = J = 0$, we obtain the values of $\widehat{rr}$ at $r = 0$ that differ by about 10 % from their true values. This conclusion appears to be also valid for general cases of distributed normal loads, even when using the approximate value $G^{(0)}$ defined via the 'zero'-

*Table 2.* Radial stresses $\widehat{rr}/p_0$ at the axis $r = 0$ of the cylinder for various numbers $(N, J)$ of unknowns found by the improved reduction and simple reduction, the infinite systems (38), (39) and (19), respectively

| | Improved reduction | | | Simple reduction | |
|---|---|---|---|---|---|
| $z/c$ | (0, 0) | (1,1) | (5,5) | (10,10) | (40,40) |
| 0·0 | 1·378 | 1·461 | 1·455 | 1·465 | 1·458 |
| 0·2 | 1·329 | 1·393 | 1·392 | 1·406 | 1·400 |
| 0·4 | 1·185 | 1·196 | 1·203 | 1·212 | 1·213 |
| 0·6 | 0·943 | 0·892 | 0·902 | 0·915 | 0·907 |
| 0·8 | 0·605 | 0·531 | 0·532 | 0·562 | 0·540 |

*Table 3.* Table of stresses $\widehat{zz}/Q$ according to the present and Filon's (in braces) analytical expressions.

| $z/c$ | $r = 0$ | $r = 0{\cdot}2a$ | $r = 0{\cdot}4a$ | $r = 0{\cdot}6a$ | $r = a$ |
|---|---|---|---|---|---|
| 0 | 0·661 (0·689) | 0·694 (0·719) | 0·796 (0·810) | 0·959 (0·962) | 1·115 ( 1·117) |
| 0·1 | 0·642 (0·673) | 0·673 (0·700) | 0·770 (0·786) | 0·935 (0·934) | 1·168 ( 1·163) |
| 0·2 | 0·592 (0·631) | 0·618 (0·652) | 0·698 (0·720) | 0·855 (0·859) | 1·344 ( 1·344) |
| 0·3 | 0·530 (0·582) | 0·548 (0·594) | 0·606 (0·637) | 0·730 (0·737) | 2·001 ( 2·022) |
| 0·4 | 0·469 (0·539) | 0·482 (0·545) | 0·522 (0·564) | 0·604 (0·617) | 1·434 ( 1·368) |
| 0·5 | 0·410 (0·503) | 0·421 (0·505) | 0·450 (0·508) | 0·488 (0·508) | 0·551 ( 0·479) |
| 0·6 | 0·341 (0·457) | 0·351 (0·457) | 0·372 (0·447) | 0·371 (0·400) | −0·325 (−0·409) |
| 0·7 | 0·250 (0·384) | 0·258 (0·361) | 0·270 (0·360) | 0·242 (0·278) | −0·867 (−1·056) |
| 0·8 | 0·142 (0·270) | 0·147 (0·272) | 0·151 (0·244) | 0·113 (0·152) | −0·184 (−0·357) |
| 0·9 | 0·043 (0·145) | 0·045 (0·141) | 0·047 (0·121) | 0·029 (0·060) | −0·134 (−0·134) |
| 1·0 | −0·001 (0·000) | 0·000 (0·000) | 0·000 (0·000) | 0·000 (0·000) | 0·012 ( 0·000) |

order Equation (39). Thus, for a rough estimate of the stresses inside the cylinder we can use the 'engineering' analytical expressions (45) with $G^{(0)}$, putting aside the finite Fourier series.

Let us now turn to the Filon example of discontinuous tangential stress (4) with $c_1 = c/3$, $c_2 = 2c/3$. The numerical values for the stresses are given in Table 3 and Table 4 in dimensionless form divided by $Q = 2cS/3a$, the uniform tension at the flat ends which would produce a pull equal to that due to the shear $S$.

The numerical results tabulated above are illustrated by curves presented in Figures 2–4. Figure 2 shows that radial tensions of order $0{\cdot}2Q$ near the middle section of the cylinder are changed to pressures after passing the ring of tangential loading. It is important to note that Filon's solution (9) cannot provide the correct values of $\widehat{rr}$ in the region $0{\cdot}75c < z < c$ because of the choice $\kappa_n$ in the system of trigonometric functions in the Fourier series. On the other hand, the plots for axial $\widehat{zz}$ and tangential $\widehat{rz}$ stresses presented in Figures 3 and 4 show a reasonable agreement between our solution and Filon's. Filon omitted the value $r = 0{\cdot}8a$, because all his series converge in this case inconveniently slowly, and no methods

*Table 4.* Table of stresses $\widehat{rz}/Q$ according to the present and Filon's (in braces) analytical expressions.

| $z/c$ | $r = 0$ | $r = 0{\cdot}2a$ | $r = 0{\cdot}4a$ | $r = 0{\cdot}6a$ | $r = a$ |
|---|---|---|---|---|---|
| 0 | 0·000 (0·000) | 0·000 (0·000) | 0·000 (0·000) | 0·000 (0·000) | 0·000 (0·000) |
| 0·1 | 0·000 (0·000) | 0·025 (0·021) | 0·057 (0·051) | 0·095 (0·089) | 0·000 (0·000) |
| 0·2 | 0·000 (0·000) | 0·040 (0·033) | 0·093 (0·082) | 0·168 (0·155) | 0·000 (0·000) |
| 0·3 | 0·000 (0·000) | 0·042 (0·033) | 0·098 (0·081) | 0·184 (0·165) | 0·000 (0·000) |
| 0·4 | 0·000 (0·000) | 0·038 (0·026) | 0·085 (0·063) | 0·157 (0·132) | 0·955 (0·955) |
| 0·5 | 0·000 (0·000) | 0·039 (0·025) | 0·082 (0·057) | 0·149 (0·119) | 0·955 (0·955) |
| 0·6 | 0·000 (0·000) | 0·051 (0·037) | 0·105 (0·081) | 0·178 (0·149) | 0·955 (0·955) |
| 0·7 | 0·000 (0·000) | 0·066 (0·058) | 0·138 (0·123) | 0·224 (0·205) | 0·000 (0·000) |
| 0·8 | 0·000 (0·000) | 0·071 (0·077) | 0·146 (0·156) | 0·221 (0·231) | 0·000 (0·000) |
| 0·9 | 0·000 (0·000) | 0·051 (0·089) | 0·105 (0·170) | 0·147 (0·219) | 0·000 (0·000) |
| 1·0 | 0·000 (0·000) | 0·000 (0·092) | 0·000 (0·172) | 0·000 (0·210) | 0·000 (0·000) |

of approximation were available. In our calculations, Figure 4*b* clearly demonstrates how the curve for the tangential stress at $r = 0{\cdot}6a$ with a double hump transforms while these two humps rise and approach each other (see data for $r = 0{\cdot}8a$) and ultimately turning into the given rectangle.

The axial $\widehat{zz}$ stress at the curved surface of the cylinder $r = a$ reveals a tendency of unlimited increase or decrease near the points of discontinuity of the applied tangential stress, as it was analytically predicted above.

Turning to the displacements $u(r, z)$ and $w(r, z)$, we observe that Filon found (see his results in braces in Tables 5 and 6) that they are very much less than the surface contraction $u_Q(a)$ and total end elongation $w_Q(c)$ (in fact, never exceeding 60 per cent) of the same cylinder under a uniform tension $Q$ over its plane ends, with

$$u_Q(r) = -\frac{Q\sigma}{E} r\,, \qquad w_Q(z) = \frac{Q}{E} z\,. \tag{56}$$

Filon gave no explanation of this discrepancy and noted that the correction (in the somewhat extreme case considered it may amount to as much as 30 per cent) should be applied to the readings of the extensometer which usually measures the relative displacement of two neighbouring points at the outer curved surface of the cylinder.

Our calculations, however, provide other data in Tables 5 and 6 that are in much better correspondence with approximate ones according to Equations (56) near the unloaded region near the middle of the cylinder. It appears that Filon [1] somewhere missed the factor 2 in his calculations in Sections 7–9, but I am unable to reveal the source of that discrepancy. Therefore, Diagrams 1 and 2 in [1], showing the distortion of the cross-sections under tangential surface loading, seem to be incorrect.

## 5. Conclusion

Looking back upon the proposed improved version of the superposition method and some results obtained by it, we see that it provides a direct and powerful algorithm for solving

*Table 5.* Table of displacements $u/u_Q(a)$ according to the present and Filon's (in braces) analytical expressions.

| $z/c$ | $r = 0$ | $r = 0{\cdot}2a$ | $r = 0{\cdot}4a$ | $r = 0{\cdot}6a$ | $r = a$ |
|---|---|---|---|---|---|
| 0·0 | 0·000 (0·000) | 0·090 (·045) | 0·213 (0·037) | 0·404 (0·134) | 1·056 (0·579) |
| 0·1 | 0·000 (0·000) | 0·088 (·039) | 0·203 (0·029) | 0·381 (0·112) | 1·022 (0·549) |
| 0·2 | 0·000 (0·000) | 0·086 (·026) | 0·188 (0·017) | 0·329 (0·065) | 0·932 (0·458) |
| 0·3 | 0·000 (0·000) | 0·096 (·020) | 0·196 (0·028) | 0·307 (0·045) | 0·777 (0·185) |
| 0·4 | 0·000 (0·000) | 0·125 (·031) | 0·252 (0·086) | 0·380 (0·121) | 0·821 (0·371) |
| 0·5 | 0·000 (0·000) | 0·170 (·057) | 0·347 (0·177) | 0·538 (0·273) | 0·955 (0·496) |
| 0·6 | 0·000 (0·000) | 0·215 (·084) | 0·445 (0·257) | 0·700 (0·413) | 1·100 (0·586) |
| 0·7 | 0·000 (0·000) | 0·250 (·093) | 0·511 (0·280) | 0·789 (0·445) | 1·164 (0·591) |
| 0·8 | 0·000 (0·000) | 0·275 (·079) | 0·547 (0·231) | 0·805 (0·353) | 1·064 (0·374) |
| 0·9 | 0·000 (0·000) | 0·313 (·045) | 0·601 (0·128) | 0·838 (0·188) | 1·067 (0·181) |
| 1·0 | 0·000 (0·000) | 0·401 (·000) | 0·752 (0·000) | 1·001 (0·000) | 1·136 (0·000) |

*Table 6.* Table of displacements $w/w_Q(c)$ according to the present and Filon's (in braces) analytical expressions.

| $z/c$ | $r = 0$ | $r = 0{\cdot}2a$ | $r = 0{\cdot}4a$ | $r = {\cdot}6a$ | $r = a$ |
|---|---|---|---|---|---|
| 0·0 | 0·000 (0·000) | 0·000 (0·000) | 0·000 (0·000) | 0·000 (0·000) | 0·000 (0·000) |
| 0·1 | 0·103 (0·057) | 0·107 (0·060) | 0·118 (0·070) | 0·136 (0·087) | 0·161 (0·110) |
| 0·2 | 0·203 (0·111) | 0·210 (0·117) | 0·230 (0·135) | 0·266 (0·167) | 0·331 (0·229) |
| 0·3 | 0·299 (0·162) | 0·308 (0·169) | 0·335 (0·192) | 0·384 (0·237) | 0·535 (0·382) |
| 0·4 | 0·391 (0·211) | 0·401 (0·219) | 0·433 (0·245) | 0·491 (0·295) | 0·800 (0·592) |
| 0·5 | 0·481 (0·260) | 0·493 (0·268) | 0·527 (0·295) | 0·591 (0·347) | 0·942 (0·671) |
| 0·6 | 0·571 (0·309) | 0·583 (0·317) | 0·620 (0·344) | 0·687 (0·395) | 1·010 (0·688) |
| 0·7 | 0·657 (0·355) | 0·670 (0·364) | 0·710 (0·390) | 0·777 (0·437) | 0·955 (0·574) |
| 0·8 | 0·737 (0·393) | 0·751 (0·402) | 0·791 (0·427) | 0·855 (0·467) | 0·965 (0·517) |
| 0·9 | 0·810 (0·418) | 0·824 (0·427) | 0·864 (0·450) | 0·923 (0·482) | 1·011 (0·497) |
| 1·0 | 0·883 (0·427) | 0·896 (0·435) | 0·933 (0·458) | 0·986 (0·487) | 1·062 (0·492) |

complicated axisymmetric biharmonic problems for the elastic finite cylinder. The method permits one to obtain accurate numerical results, using only a few terms in the series on the complete systems of trigonometric and Bessel functions, even for the case of discontinuous tangential load. With sufficient perseverance, the method of superposition permits to address the more complicated axisymmetric problem of a cylindrical specimen with an abrupt change of radius in a testing machine (Figure 1). The algebraic work involved is rather cumbersome, but the final formulae are believed to be rather simple for numerical evaluation. One can argue, of course, that any currently available finite-element or boundary-element commercial package can solve the given problem with significantly less effort. Nevertheless, the present analytical treatment based upon the 'first principles' of the Fourier series could be useful as a

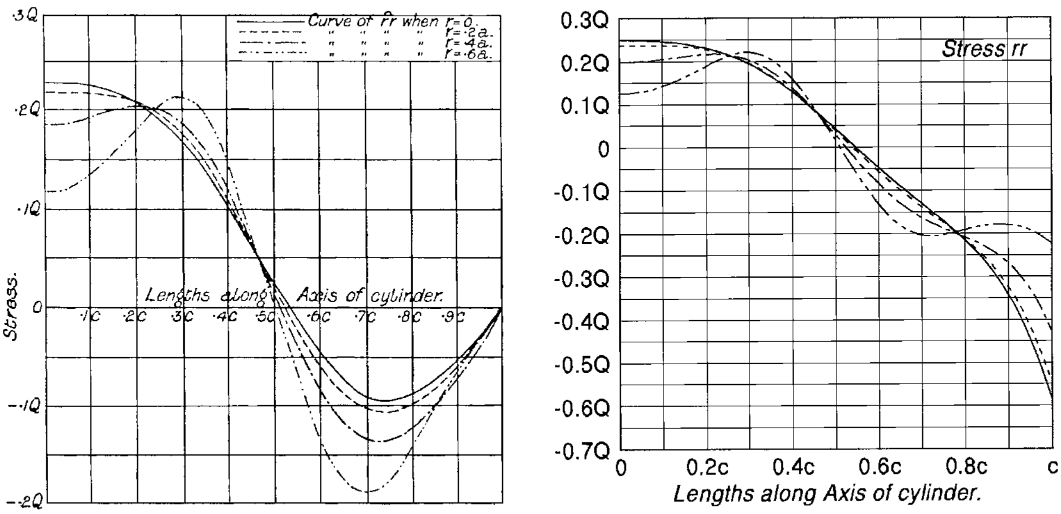


*Figure 2.* Distribution of the radial stress $\widehat{rr}$ for the cylinder under a shearing pull. Solid line – $r = 0$; short dashed line – $r = 0{\cdot}2a$; long dashed dot line – $r = 0{\cdot}4a$; long dashed double dot line – $r = 0{\cdot}6a$. (a) Filon's [1] solution; (b) present analytical solution.

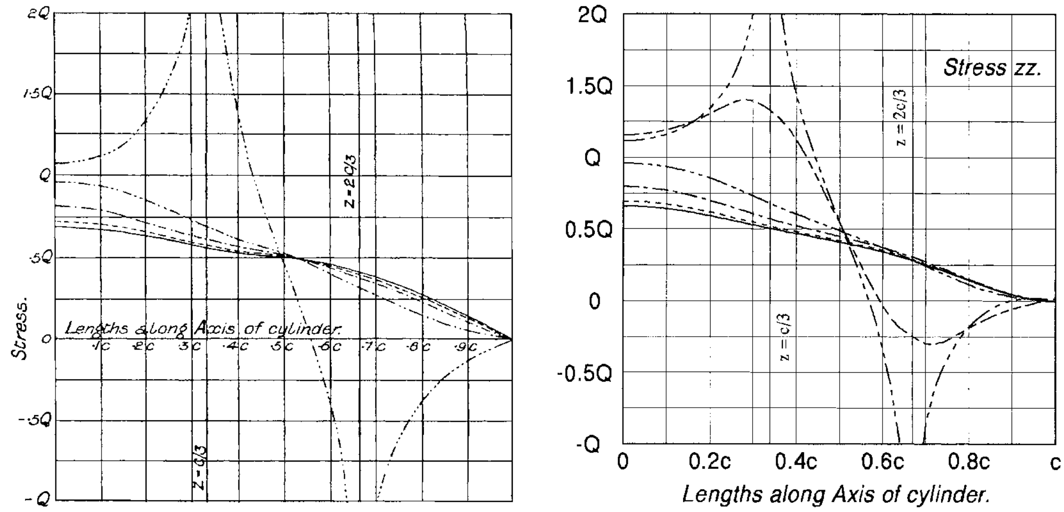


*Figure 3.* Distribution of the axial stress $\widehat{zz}$ for the cylinder under a shearing pull. Solid line – $r = 0$; short dashed line – $r = 0{\cdot}2a$; long dashed dot line – $r = 0{\cdot}4a$; long dashed double dot line – $r = 0{\cdot}6a$; long dashed triple dot line – $r = a$. (a) Filon's [1] solution; (b) present analytical solution.

benchmark example for testing the accuracy of any numerical scheme, especially near rims of the cylinder and points of discontinuity of applied loadings.

However, the second problem nominated for the competition for the *Grand Prix de Mathématiques* about the three-dimensional equilibrium of an elastic finite parallelepiped under an arbitrary system of loads nonuniformly distributed on its sides, the famous Lamé problem that was also advanced 150 years ago in his twelfth lecture [24], still remains challenging.

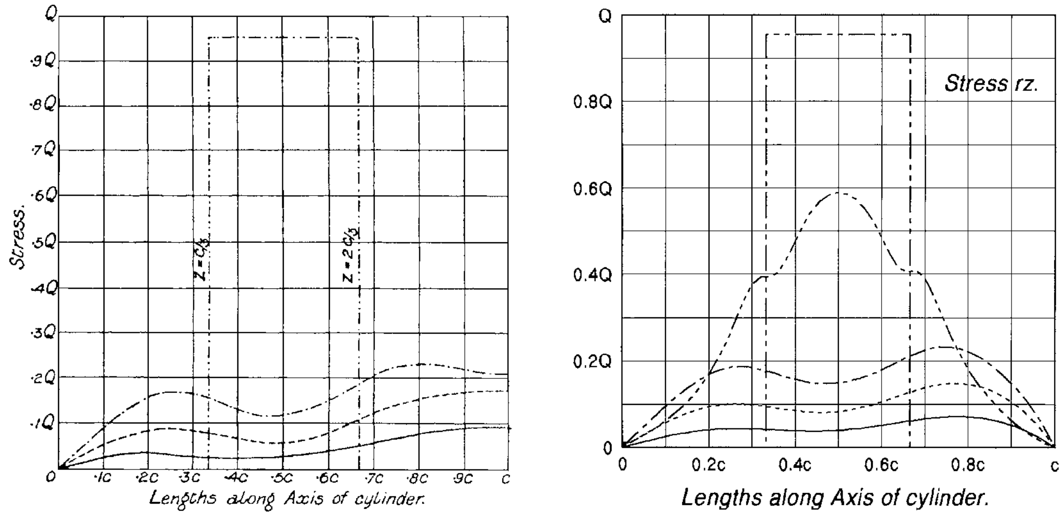


*Figure 4.* Distribution of the shear stress $\widehat{rz}$ for the cylinder under a shearing pull. Solid line – $r = 0{\cdot}2a$; short dashed line – $r = 0{\cdot}4a$; long dashed dot line – $r = 0{\cdot}6a$; long dashed triple dot line – $r = 0{\cdot}8a$ long dashed double dot line – $r = a$. (a) Filon's [1] solution; (b) present analytical solution.

## Acknowledgements

My very best thanks are due to Victor Grinchenko of the Institute of Hydromechanics (Kiev) and Andrei Ulitko of Kiev National University for their unfailing kindness in coming to my aid with suggestions and advice during a long period of mutual work. I am grateful to Alexandre Gourjii and Valerii Oliinyk of the Institute of Hydromechanics for help with the arithmetic of the paper and presenting the graphical data, respectively.

## Appendix: Fourier and Dini expansions and sums in the closed form

The following Fourier and Dini expansions [17, 23] on the complete trigonometric and Bessel functions are recorded here

$$\lambda z \frac{\sinh \lambda z}{\sinh \lambda c} + (1 - \lambda c \coth \lambda c) \frac{\cosh \lambda z}{\sinh \lambda c} = \sum_{n=1}^{\infty} (-1)^n \frac{4\lambda k_n^2}{c(k_n^2 + \lambda^2)^2} \cos k_n z \,, \tag{A1}$$

$$z = -\sum_{n=1}^{\infty} (-1)^n \frac{2}{k_n} \sin k_n z \,, \quad \frac{\sinh \lambda z}{\sinh \lambda c} = -\sum_{n=1}^{\infty} (-1)^n \frac{2k_n}{c\left(k_n^2 + \lambda^2\right)} \sin k_n z \,, \tag{A2}$$

$$z \frac{\cosh \lambda z}{\sinh \lambda c} - c \coth \lambda c \frac{\sinh \lambda z}{\sinh \lambda c} = \sum_{n=1}^{\infty} (-1)^n \frac{4\lambda k_n}{c\left(k_n^2 + \lambda^2\right)^2} \sin k_n z \,, \tag{A3}$$

$$r \frac{I_1(kr)}{I_1(ka)} + \left[ \frac{2}{k} - a \frac{I_0(ka)}{I_1(ka)} \right] \frac{I_0(kr)}{I_1(ka)} = \sum_{j=1}^{\infty} \frac{4\lambda_j^2}{a(\lambda_j^2 + k^2)^2} \frac{J_0(\lambda_j r)}{J_0(\lambda_j a)} \,, \tag{A4}$$

$$\frac{I_0(kr)}{I_1(ka)} = \frac{2}{ka} + \sum_{j=1}^{\infty} \frac{2k}{a(\lambda_j^2 + k^2)} \frac{J_0(\lambda_j r)}{J_0(\lambda_j a)} \,, \tag{A5}$$

$$a\frac{I_0(ka)}{I_1(ka)}\,\frac{I_0(kr)}{I_1(ka)} - r\frac{I_1(kr)}{I_1(ka)} = \frac{4}{k^2 a} + \sum_{j=1}^{\infty}\frac{4k^2}{a(\lambda_j^2+k^2)^2}\,\frac{J_0(\lambda_j r)}{J_0(\lambda_j a)}\,, \tag{A6}$$

with $k_n = \dfrac{n\pi}{c}$ and $\lambda_j$ is a nonzero root of the equation $J_1(\lambda_j a) = 0$, are used in the main text.

Also, the following sums

$$\sum_{n=1}^{\infty}\frac{4\lambda^2}{\left(k_n^2+\lambda^2\right)^2} = \frac{c}{\lambda}\left(\coth\lambda c + \frac{\lambda c}{\sinh^2\lambda c} - \frac{2}{\lambda c}\right)\,, \tag{A7}$$

$$\sum_{n=1}^{\infty}\frac{4k_n^2}{\left(k_n^2+\lambda^2\right)^2} = \frac{c}{\lambda}\left(\coth\lambda c - \frac{\lambda c}{\sinh^2\lambda c}\right)\,, \tag{A8}$$

$$\sum_{j=1}^{\infty}\frac{4k^2}{(\lambda_j^2+k^2)^2} = a^2\left[\frac{I_0^2(ka)}{I_1^2(ka)} - 1\right] - \frac{4}{k^2}\,, \tag{A9}$$

$$\sum_{j=1}^{\infty}\frac{4\lambda_j^2}{(\lambda_j^2+k^2)^2} = a^2\left[1 - \frac{I_0^2(ka)}{I_1^2(ka)}\right] + \frac{2aI_0(ka)}{kI_1(ka)}\,, \tag{A10}$$

which can be obtained by putting $z = c$ and $r = a$ in uniformly convergent series (57), (60), and (62).

By putting in (64) and (66) $\lambda \to 0$ and $k \to 0$, respectively, the values of sums

$$\sum_{n=1}^{\infty}\frac{1}{k_n^2} = \frac{c^2}{6}\,, \qquad \sum_{j=1}^{\infty}\frac{1}{\lambda_j^2} = \frac{a^2}{8} \tag{A11}$$

are obtained.

The asymptotic behaviour when $k \to \infty$ is

$$\frac{I_0(k)}{I_1(k)} = 1 + \frac{1}{2k} + \frac{3}{8k^2} + O\left(\frac{1}{k^3}\right)\,, \quad \frac{I_0^2(k)}{I_1^2(k)} = 1 + \frac{1}{k} + \frac{1}{k^2} + O\left(\frac{1}{k^3}\right)\,, \tag{A12}$$

which can be obtained from semi-convergent expansions of the modified Bessel functions $I_0(k)$ and $I_1(k)$.

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